JACOB'S LADDERS AND THE QUANTIZATION OF THE HARDY-LITTLEWOOD INTEGRAL

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Abstract. We use Jacob's ladders to solve the fine problem how to divide of the Hardy-Littlewood integral to equal parts, for example of magnitude $h = 6.6 \times 10^{-27}$ (the numerical value of elementary Planck quantum). The result of the paper cannot be obtained in known theories of Balasubramanian, Heath-Brown and Ivic.

1. The problem of dividing on equal parts

1.1. Let us remind the following facts. Titchmarsh-Kober-Atkinson (TKA) formula

$$\int_0^\infty Z^2(t)e^{-2\delta t}\mathrm{d}t = \frac{c-\ln(4\pi\delta)}{2\sin\delta} + \sum_{n=1}^N c_n\delta^n + \mathcal{O}(\delta^{N+1}) \tag{1.1}$$

(see [10], p. 141) remained as an isolated result for a period of 56 years until we have discovered the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t)e^{-\frac{2}{x(T)}t}\mathrm{d}t = \int_0^T Z^2(t)\mathrm{d}t \tag{1.2}$$

(see [6]) in which the essence of the TKA formula is encoded. Namely, we have shown in [6] that the following almost exact formula for the Hardy-Littlewood integral takes place

$$\int_0^T Z^2(t)\mathrm{d}t = \frac{\varphi(T)}{2}\ln\frac{\varphi(T)}{2} + (c-2\pi)\frac{\varphi(T)}{2} + c_0 + \mathcal{O}\left(\frac{\ln T}{T}\right), \tag{1.3}$$

where $\varphi(T)$ is the Jacob's ladder (a solution of the nonlinear integral equation (1.2)).

Remark 1. Our formula (1.3) for the Hardy-Littlewood integral has been obtained after the time period of 90 years since this integral appeared in 1918 (see [3], pp. 122, 151-156).

Remark 2. Let us remind that

(A) The Good's Ω-theorem (see [2]) implies for the Balasubramanian formula

$$\int_0^T Z^2(t)\mathrm{d}t = T\ln T + (2c-1-\ln 2\pi)T + R(T),\ R(T) = \mathcal{O}(T^{1/3+\epsilon}) \tag{1.4}$$

(see [1]) that

$$\limsup_{T\to\infty} |R(T)| = +\infty.$$

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(B) The error term in (1.3) tends to zero as T goes to infinity, namely

$$\lim_{T\to\infty} r(T)=0,\ r(T)=\mathcal{O}\left(\frac{\ln T}{T}\right),$$

i.e. our formula is almost exact (see [6]).

1.2. In this paper I consider the problem concerning the solid of revolution corresponding to the graph of the function $Z(t),\ t\in[T_0,+\infty)$, where $0<T_0$ is a sufficiently big number.

Problem. To divide this solid of revolution on parts of equal volumes.

We obtain, for example, from our formula (1.3) that there exists a sequence $\{\hat{T}_\nu\}_{\nu=\nu_0}^\infty$, for which

$$\pi\int_{\hat{T}_\nu}^{\hat{T}_{\nu+1}} Z^2(t)\mathrm{d}t=6.6\times 10^{-27}, \tag{1.5}$$

$$\hat{T}_{\nu+1}-\hat{T}_\nu\sim\frac{6.6\times 10^{-27}}{\pi\ln\hat{T}_\nu\tan[\alpha(\hat{T}_\nu,\hat{T}_{\nu+1})]},\ \nu\to\infty,$$

where $h=6.6\times 10^{-27}\mathrm{erg}\cdot\mathrm{sec}$ is elementary Planck quantum.

Remark 3. It us quite evident that the quantization rule (1.5) cannot be obtained by methods of Balasubramanian, Heath-Brown and Ivic (see, for example, [4]).

This paper is a continuation of the series of papers [6]-[8].

2. Main result

The following theorem holds true

Theorem 1. *Let $0<\delta<\Delta$ where δ is an arbitrarily small and Δ is an arbitrarily big number. Then for every $\omega\in[\delta,\Delta]$ and every Jacob's ladder $\varphi(T)$ there is the sequence*

$$\{T_\nu(\omega,\varphi)\}_{\nu=\nu_0}^\infty,\quad T_\nu(\omega,\varphi)=T_\nu(\omega)$$

for which

$$\int_{T_\nu(\omega)}^{T_{\nu+1}(\omega)} Z^2(t)\mathrm{d}t=\omega, \tag{2.1}$$

$$T_{\nu+1}(\omega)-T_\nu(\omega)=\frac{\omega+\mathcal{O}\left(\frac{\ln T_\nu}{T_\nu}\right)}{\left(\ln\frac{\varphi(T_\nu)}{2}-a\right)\tan[\alpha(T_\nu,T_{\nu+1})]}, \tag{2.2}$$

where $0<\nu_0(\omega,\varphi)$ is a sufficiently big number, $a=\ln 2\pi-1-c$ and $\alpha=\alpha(T_\nu,T_{\nu+1})$ is he angle of the chord binding the points

$$\left[T_\nu,\frac{1}{2}\varphi[T_\nu(\omega)]\right],\quad\left[T_{\nu+1},\frac{1}{2}\varphi[T_{\nu+1}(\omega)]\right]$$

of the curve $y=\frac{1}{2}\varphi(T)$.

We obtain from our Theorem 1

Corollary 1.

$$\int_{T_\nu(\omega)}^{T_{\nu+1}(\omega)} Z^2(t)\mathrm{d}t = \tag{2.3}$$
$$[T_{\nu+1}(\omega)-T_\nu(\omega)]\ln\left(e^{-a}\frac{\varphi(T_\nu)}{2}\right)\tan[\alpha(T_\nu,T_{\nu+1})]+\mathcal{O}\left(\frac{\ln T_\nu}{T_\nu}\right).$$

Let us remind the formula (see [7], (2.1))

$$\int_T^{T+U} Z^2(t)\mathrm{d}t = U\ln\left(e^{-a}\frac{\varphi(T)}{2}\right)\tan[\alpha(T,U)]+\mathcal{O}\left(\frac{1}{T^{1/3-4\epsilon}}\right). \tag{2.4}$$

Remark 4. In the case of collection of sequences $\{T_\nu(\omega)\}$ we obtain the essential improvement

$$\mathcal{O}\left(\frac{1}{T^{1/3-4\epsilon}}\right)\to\mathcal{O}\left(\frac{\ln T_\nu}{T_\nu}\right)$$

(see (2.3) of the remainder term in the formula (2.4).

Since

$$\sum_{k=1}^{N}\int_{T_{\nu+k-1}(\omega)}^{T_{\nu+k}(\omega)} Z^2(t)\mathrm{d}t = N\omega$$

then, in the case

$$T_{\nu+N_0}(\omega)-T_\nu(\omega)\sim U_0 = T^{1/3+2\epsilon},$$

we have (see (1.4))

$$N_0\sim\frac{1}{\omega}U_0\ln T_\nu(\omega),\ \frac{U_0}{N_0}\sim\frac{\omega}{\ln T_\nu(\omega)}.$$

Then we obtain by our Theorem 1

Corollary 2. The following asymptotic formula takes place

$$\frac{1}{N_0}\sum_{k=1}^{N_0}\{T_{\nu+k}(\omega)-T_{\nu+k-1}(\omega)\}\sim\frac{\omega}{\ln T_\nu(\omega)} \tag{2.5}$$

for arithmetic mean values of $T_{\nu+k}(\omega)-T_{\nu+k-1}(\omega)$, $k=1,2,\dots,N_0$.

3. On transformation of the sequence $\{T_\nu(\omega)\}$ preserving the quantization of the Hardy-Littlewood integral

3.1. If there is a sufficiently big natural number $\bar{\nu}$ for which $\omega\bar{\nu}=T_0$ is fulfilled then from our Theorem 1 the resolution of our Problem follows. The complete resolution follows from the next theorem.

Theorem 2. *For every $\omega\in[\delta,\Delta]$, $\tau\in[0,\omega)$ and every Jacob's ladder $\varphi(T)$ there is the collection of sequences*

$$\{T_\nu(\omega,\tau;\varphi)\}_{\nu=\nu_0}^{\infty},\quad T_\nu(\omega,\tau;\varphi)=T_\nu(\omega,\tau),\ T_\nu(\omega,0)=T_\nu(\omega)$$

for which

$$\int_{T_\nu(\omega,\tau)}^{T_{\nu+1}(\omega,\tau)} Z^2(t)\mathrm{d}t=\omega, \tag{3.1}$$

$$T_{\nu+1}(\omega,\tau)-T_\nu(\omega,\tau)=\frac{\omega+\mathcal{O}\left(\frac{\ln T_\nu}{T_\nu}\right)}{\left(\ln\frac{\varphi(T_\nu)}{2}-a\right)\tan[\alpha(T_\nu,T_{\nu+1})]}, \tag{3.2}$$

where $0<\nu_0(\omega,\tau;\varphi)$ is a sufficiently big number and $\alpha=\alpha(T_\nu,T_{\nu+1})$ is the angle of the chord binding the points

$$\left[T_\nu(\omega,\tau),\frac{1}{2}\varphi(T_\nu(\omega,\tau))\right],\quad \left[T_{\nu+1}(\omega,\tau),\frac{1}{2}\varphi(T_{\nu+1}(\omega,\tau))\right]$$

of the curve $y=\frac{1}{2}\varphi(T)$.

Remark 5. From (3.1), (3.2) we get full analogies of Corollaries 1., 2. and Remark 4.

Remark 6. The quantization (1.5) follows from the choice $\{T_\nu(\omega,\tau)\}$ with $\omega=\frac{h}{\pi},\ \bar{\nu}\omega+\tau=T_0$.

3.2. Let us remind that for Gram's sequence $\{t_\nu\}$ we have (see [9])

$$t_{\nu+1}-t_\nu\sim\frac{2\pi}{\ln t_\nu}, \tag{3.3}$$

and for the collection of sequences $\{\bar{t}_\nu(\bar{\tau})\},\ \bar{\tau}\in[-\pi,\pi]$ defined in our paper [5] we have the analogue of (3.3)

$$\bar{t}_{\nu+1}(\bar{\tau})-\bar{t}_\nu(\bar{\tau})\sim\frac{2\pi}{\ln t_\nu(\bar{\tau})}.$$

Remark 7. Under the transformations

$$t_\nu\to T_\nu(\omega);\ T_\nu(\omega,\tau)$$

the individual property (3.3) is transformed to an analogous property of arithmetic means (see (2.5) and Remark 5).

4. Proof of Theorems 1., 2.

4.1. By (1.3) we have

$$\int_0^T Z^2(t)\mathrm{d}t=F[\varphi(T)]+\mathcal{O}\left(\frac{\ln T}{T}\right),\ T\geq T^{(1)}[\varphi], \tag{4.1}$$

$$F(y)=\frac{y}{2}\ln\frac{y}{2}+(c-\ln 2\pi)\frac{y}{2}+c_0,\ F'(y)=\frac{1}{2}\ln\frac{y}{2}-\frac{a}{2},\ F''(y)=\frac{1}{2y}. \tag{4.2}$$

Since the continuous function

$$F[\varphi(T)]+\mathcal{O}\left(\frac{\ln T}{T}\right),\ T\geq T^{(1)}$$

is increasing, there is a root $T_\nu(\omega,\varphi)$ of the equation

$$F[\varphi(T)]+\mathcal{O}\left(\frac{\ln T}{T}\right)=\omega\nu,\quad \nu\geq\nu_0,$$

where $\nu_0=\nu_0(\omega,\varphi)$ is a sufficiently big number. Thus, the sequence $\{T_\nu(\omega;\varphi)\}_{\nu=\nu_0}^\infty$ is constructed by equation

$$F[\varphi(T_\nu)]+\mathcal{O}\left(\frac{\ln T_\nu}{T_\nu}\right)=\omega\nu. \tag{4.3}$$

From (4.1) by (4.3) we obtain

$$\int_0^{T_\nu(\omega;\varphi)} Z^2(t)\mathrm{d}t = \omega\nu \ \Rightarrow \ \int_{T_\nu(\omega;\varphi)}^{T_{\nu+1}(\omega;\varphi)} Z^2(t)\mathrm{d}t = \omega,$$

i.e. (2.1).

4.2. We have by (4.2), (4.3)

$$\omega + \mathcal{O}\left(\frac{\ln T_\nu}{T_\nu}\right) = F[\varphi(T_{\nu+1})] - F[\varphi(T_\nu)] =$$
$$\left(\frac{1}{2}\ln\frac{d}{2} - \frac{a}{2}\right)[\varphi(T_{\nu+1}) - \varphi(T_\nu)], \ d \in (\varphi(T_\nu), \varphi(T_{\nu+1})),$$

i.e. (see [6], (5.2); $1.9T < \varphi(T) < 2T$)

$$\varphi(T_{\nu+1}) - \varphi(T_\nu) = \mathcal{O}\left(\frac{1}{\ln T\nu}\right). \tag{4.4}$$

Next we have

$$F[\varphi(T_{\nu+1})] - F[\varphi(T_\nu)] = \left(\frac{1}{2}\ln\frac{\varphi(T_\nu)}{2} - \frac{a}{2}\right)[\varphi(T_{\nu+1}) - \varphi(T_\nu)] +$$
$$\mathcal{O}\left\{\frac{(\varphi(T_{\nu+1}) - \varphi(T_\nu))^2}{T}\right\} + \mathcal{O}\left(\frac{\ln T_\nu}{T_\nu}\right) = \omega$$

by (4.2), (4.3), i.e. (see (4.4))

$$\frac{1}{2}[\varphi(T_{\nu+1}) - \varphi(T_\nu)]\left(\ln\frac{\varphi(T_\nu)}{2} - a\right) = \omega + \mathcal{O}\left(\frac{\ln T_\nu}{T_\nu}\right) + \mathcal{O}\left(\frac{1}{T_\nu \ln^2 T_\nu}\right),$$

from which the formula (2.2) follows.

Remark 8. Proof of Theorem 2 is similar to the proof of Theorem 1.

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